

Scaling Exponents for Kinetic Roughening in Higher Dimensions

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Received December 31, 1992; final January 25, 1993

We discuss the results of extensive numerical simulations in order to estimate the scaling exponents associated with kinetic roughening in higher dimensions, up to $d=7+1$. To this end, we study the restricted solid-on-solid growth model, for which we employ a novel fitting ansatz for the spatially averaged height correlation function $\bar{G}(t) \sim t^{2\beta}$ to estimate the scaling exponent β . Using this method, we present a quantitative determination of β in $d=3+1$ and $4+1$ dimensions. To check the consistency of these results, we also compute the interface width and determine β and χ from it independently. Our results are in disagreement with all existing theories and conjectures, but in four dimensions they are in good agreement with recent simulations of Forrest and Tang for a different growth model. Above five dimensions, we use the time dependence of the width to obtain lower bound estimates for β . Within the accuracy of our data, we find no indication of an upper critical dimension up to $d=7+1$.

KEY WORDS: Surface growth; kinetic roughening; solid-on-solid model.

1. INTRODUCTION

Dynamical behavior of interfaces in random media constitutes a fundamental problem in statistical mechanics,⁽¹⁾ with applications to, e.g., fluid mechanics,^(2,3) surface growth,^(4,5) polymer physics,⁽⁶⁾ and magnetic flux lines in superconductors.⁽⁷⁾ For a very large class of physical systems, the essential physics associated with the dynamics of such objects can be

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described in the continuum language by the Kardar–Parisi–Zhang (KPZ) equation^(4,8):

$$\frac{\partial h}{\partial t} = v\nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\mathbf{r}, t) + \mu \quad (1)$$

In particular, for many simple ballistic growth models Eq. (1) is now widely accepted to be the relevant mapping in the continuum limit.^(1,9) For these models, Eq. (1) has a particularly transparent physical interpretation: the height variable $h(\mathbf{r}, t)$ describes a growing interface on a ($d_s = d - 1$)-dimensional hyperplane, μ is a constant driving force, and v and λ describe the effects of surface tension and lateral growth velocity, respectively. The random variable η is Gaussian and satisfies

$$\langle \eta(\mathbf{r}, t) \eta(\mathbf{r}', t') \rangle = 2D \delta^{d_s}(\mathbf{r} - \mathbf{r}') \delta(t - t') \quad (2)$$

with D describing local variations in the deposition rate.

The properties of the KPZ equation are usually described through the first two moments of the associated probability distribution function of the variables $\{h(\mathbf{r}, t)\}$. Namely, the average interface height $\bar{h}(t) \equiv \langle h(\mathbf{r}, t) \rangle$ grows linearly in time, while the surface width $w(L, t)$, which is defined as the standard deviation of heights

$$w^2(L, t) = \int d\mathbf{r} [h(\mathbf{r}, t) - \bar{h}(t)]^2 / L^{d_s} \quad (3)$$

displays nontrivial scaling behavior as a function of time and the linear system size,^(4,8,10)

$$w(L, t) \sim L^\chi f\left(\frac{t}{L^z}\right) \quad (4)$$

The scaling function $f(x) \propto x^\beta$ for $x \ll 1$, with $\beta \equiv \chi/z$, and becomes constant for $x \gg 1$. The KPZ equation also satisfies an exact invariance under a “Galilean” transformation, which leads to the identity $\chi + z = 2$,^(2,54,8) and leaves only one independent scaling exponent to be determined.

The scaling indices β , χ , and z for the KPZ equation are exactly known in $d = 1 + 1$ dimensions, where the fluctuation-dissipation relation and Galilean invariance amalgamate to yield $\beta(2) = 1/3$, $\chi(2) = 1/2$, and $z(2) = 3/2$.^(1,2,4,8) These numbers have also been consistently obtained from direct integrations of the (discretized) KPZ equation^(11,12) and simulations of ballistic growth models,^(1,9,13–15) implying universal behavior as embodied by the KPZ equation. In addition to the exponents, the scaling

function has recently been calculated numerically using self-consistent mode coupling equations.⁽¹⁶⁾ The concept of universality between different models has further been strengthened by recent calculations, where certain amplitude ratios have been shown to be universal for several ballistic growth models.⁽¹⁷⁻¹⁹⁾

In higher dimensions, where no fluctuation-dissipation relation exists to fix χ and where the dynamical renormalization group efforts at the strong coupling fixed point fail, exact results are few. Only the weak coupling (ideal interface) case where $\lambda = 0$ is exactly known,^(20,21) together with the dynamical exponent $z_c = 2$ at the roughening transition of the KPZ equation to this smooth phase above $d > 2 + 1$.⁽²²⁾ Not surprisingly, then, a wealth of approximate analytic schemes^(1,23-25) and numerical simulations^(1,9,11,12,14,15,26,27) have been employed to evaluate the scaling exponents and their behavior in higher dimensions. However, the situation remains unsettled. In particular, both recent functional renormalization group calculations by Halpin-Healy⁽²³⁾ and $1/d$ expansions results of Cook and Derrida⁽²⁴⁾ indicate the existence of an *upper critical dimension* d_u above which the strong-coupling phase vanishes and $z = 2$, $\beta = \chi = 0$. In Halpin-Healy's theory, $d_u = 4 + 1$.

Numerical simulations of various realizations of the KPZ equation also give contradictory results.⁽¹⁾ Direct attempts to solve the discretized KPZ equation have apparently been riddled by stability problems, and rather different results have been reported.^(11,12) Neither have the results from simulations of directed polymers in random media^(1,26,27) nor various ballistic growth models^(1,9,14) converged. One of the most serious recent attempts to determine the scaling exponents has been the conjecture of Kim *et al.*,⁽¹⁴⁾ which was based on numerical results obtained for the restricted solid-on-solid growth (GRSOS) model. According to this conjecture,

$$\beta(d) = \frac{1}{d+1}, \quad \chi(d) = \frac{2}{d+2}, \quad z(d) = \frac{2(d+1)}{d+2} \quad (5)$$

which is correct at $d = 0 + 1$ and $1 + 1$, and implies no finite upper critical dimension in the problem. However, this result has been seriously challenged by large-scale simulations of Forrest *et al.*⁽⁹⁾ on a hypercube stacking model, from which they obtained $\beta(3) = 0.240(1)$ and $\beta(4) = 0.180(5)$, which are slightly but distinctly lower than the predictions of Eq. (5). To add to the confusion, Kim *et al.*⁽²⁷⁾ have recently done additional studies of the directed polymer problem, with results in accordance with Eq. (5) at three and four dimensions.

The purpose of the present work is to try to answer some of the open

questions regarding the KPZ equation in higher dimensions. The first and the most basic one concerns the actual values of the scaling exponents. A related question is the controversy between the results of Forrest *et al.*⁽⁹⁾ and the conjecture of Eq. (5). This is particularly important in addressing the question of universality between ballistic growth models beyond two dimensions. A third question concerns the existence of an upper critical dimension. The only numerical work which to our knowledge has addressed this is the directed polymer simulation of Renz⁽²⁶⁾ (see also ref. 1), which shows no indication of d_u up to six dimensions.

To address all these questions, we have undertaken new numerical simulations of the GRSOS model in higher dimensions, up to $d=7+1$. In this work, we shall concentrate on dimensions $d \geq 3+1$, where the original work of Kim *et al.*⁽¹⁴⁾ was most seriously affected by finite-size effects, and where possible deviations from Eq. (5) could be most clearly seen. In particular, we have developed a new way of extracting the growth exponents using a *fitting ansatz*⁽¹⁵⁾ for the equal-time height correlation function $G(\mathbf{r}, t)$. This fitting ansatz is first tested in two dimensions, and then applied to extensive numerical simulations in $d=3+1$, where we have obtained β with great accuracy. Our most accurate estimate, $\beta(4)=0.180(2)$ is well below the prediction of the conjecture (5), but in remarkably accurate agreement with the results of Forrest *et al.*,⁽⁹⁾ indicating universality between these two growth models. The numerical value of $\beta(4)$ is further supported by standard determinations of $\beta(4)$ and $\chi(4)$ from the surface width $w(L, t)$, for a variety of system sizes. Furthermore, we have used the surface width in $d=4+1$ to similarly obtain an estimate for $\beta(5)$ and $\chi(5)$. Above five dimensions, where finite-size effects become prohibitively difficult to overcome, we have simply used the slope of the growing surface width to obtain at least lower bound estimates for β . Since relatively well-defined power laws are found in each case, we find no indication of d_u up to eight dimensions. Finally, we also discuss briefly how our fitting ansatz can be used to study the question of amplitude universality.

2. THE GROWTH MODEL

We define the restricted solid-on-solid growth (GRSOS) model as a ballistic growth model, where particles of height unity are randomly deposited on an initially flat surface.⁽¹⁴⁾ The essential feature of the model is the condition that for local growth to occur, the height differences between all the $2d_s$ nearest neighbor columns (on a hypercubic lattice) must satisfy the condition $|Ah| \leq 1$. If this condition is violated, the deposition attempt is aborted; however, no other desorption events can take

place. Thus, in the computer simulations each Monte Carlo step (MCS) consists of L^d random deposition attempts. We note that neither parallel nor sublattice growth algorithms have been used in the present study.

Extensive studies of the GRSOS model have shown⁽¹⁴⁾ that the condition $|\Delta h| \leq 1$ leads to a very rapid approach of the surface width and other relevant quantities toward nontrivial scaling behavior, which in two dimensions recovers the exact KPZ exponents. Moreover, recent explicit numerical determinations of λ in two and three dimensions^(13,17,18) and studies of universal amplitude ratios in $d=1+1$ ^(17,19) have further solidified the relation of the GRSOS model to the KPZ equation. Furthermore, similarly to the closely related hypercube stacking model,⁽⁹⁾ the GRSOS interface can be described in terms of a waiting time distribution, which corresponds to the discrete directed polymer mapping of the KPZ equation. Thus, even beyond two dimensions, we expect the GRSOS model to be able to describe the strong-coupling behavior of the KPZ equation.

3. A NEW METHOD OF EXTRACTING THE SCALING EXPONENTS

As mentioned in the Introduction, there exists a body of numerical work which has aimed at the quantitative evaluation of the scaling exponents. This can in principle be done in at least three different ways: (i) trying to solve a discretized version of the KPZ equation directly by numerical iteration, (ii) simulating the directed polymer mapping of the KPZ equation at zero temperature, or (iii) simulating a discrete growth model which belongs to the same universality class as the KPZ equation. Most attempts to date have been based on (iii)⁽¹⁾; however, some models are apparently plagued by severe crossover and finite-size effects, which in part have contributed to the discrepancies reported in the literature. A related problem in the existing numerical studies lies in the actual methods used to extract the exponents. In almost all previous numerical works⁽¹⁾ β and χ have been determined using the relations $w(t) \sim t^\beta$ and $w(L) \sim L^\chi$. However, it has been shown⁽¹⁴⁾ that the slope of the time-dependent surface width tends to *underestimate* β due to a finite-size correction. Instead, the spatially averaged correlation function

$$\bar{G}(t) \equiv \langle G(\mathbf{r}, t) \rangle_r \sim t^{2\beta} \quad (6)$$

where

$$G(\mathbf{r}, t) \equiv \langle [h(\mathbf{x} + \mathbf{r}, t) - h(\mathbf{x}, t)]^2 \rangle_x \quad (7)$$

and the average $\langle \cdot \rangle_r$, taken in the *asymptotic regime* of $r \gg t^{1/2}$, has been shown to give very good estimates of β even for relatively small system

sizes. This is due to the fact that $\bar{G}(t)$ averaged over larger distances only lacks short-wavelength components, which are irrelevant for the asymptotic behavior. There is an additional advantage of using the correlation function, because asymptotically

$$G(r, t) \sim \begin{cases} r^{2\chi} & \text{for } r \ll t^{1/z} \\ t^{2\beta} & \text{for } r \gg t^{1/z} \end{cases} \quad (8)$$

Thus, it is also possible to obtain independent estimates of χ and z directly from the same set of data, as we will discuss below.

However, when using $\bar{G}(t)$ there exists a fundamental problem in determining the onset of the asymptotic regime which is to be used for obtaining $\bar{G}(t)$. In practice this means introducing a time-dependent cutoff parameter, which is somewhat arbitrary. To overcome this problem, and to utilize fully the information contained in the correlation function, we have developed a novel *fitting ansatz* for the correlation function (7) as⁽¹⁵⁾

$$\hat{G}(r, t) = a(t) \{ \tanh[b(t)^{1/x} r^{2\hat{\chi}(t)/x}] \}^x \quad (9)$$

where $a(t)$, $b(t)$, and $\hat{\chi}(t)$ are fitting parameters, and x is fixed. This functional form is motivated by the limiting behavior of the correlation function. Namely, for each fixed time $1 \ll t \ll L^z$, $\hat{G}(r, t) = a(t)b(t)r^{2\hat{\chi}(t)}$ for $r \rightarrow 0$, while $\hat{G}(r, t) = a(t)$ for $r \rightarrow \infty$. Thus, after fixing x , we can use Eq. (9) to fit *the whole function* $G(r, t)$. First, this gives us

$$\bar{G}(t) \approx a(t) \sim r^{2\beta} \quad (10)$$

from which β can be immediately obtained without a cutoff parameter. Moreover, the fitting gives us an independent estimate for $\hat{\chi} \approx \chi$, as can be seen from Eq. (8). The third scaling exponent $z \approx \hat{z}$ can also be obtained by defining a radius⁽¹⁴⁾

$$r_c(t) \sim t^{1/\hat{z}} \quad (11)$$

through the condition that $\hat{G}(r_c(t), t) = ca(t) \approx c\bar{G}(t)$, where $c < 1$ is fixed. By carefully studying the dependence of \hat{z} on c , it should be possible to estimate z rather accurately, if the ansatz (9) is reliable. We note that Eq. (9) offers by no means a unique choice for the ansatz. Nevertheless, it is in a convenient form for fitting the correlation radius $r_c(t)$ accurately by varying x , which was also explicitly tested not to affect the value of $a(t)$ as obtained from the fits. However, the influence of other functional forms was not studied here.

4. RESULTS

4.1. $d = 1 + 1$

To test the method of fitting the correlation function using the ansatz (9), we first consider the two-dimensional case, for which $\beta(2) = 1/3$ and $\chi(2) = 1/2$ are exact. To this end, we chose to simulate a $L = 3000$ system, which is relatively small, for which we determined $G(r, t)$ accurately by averaging over 3000 independent runs, up to 800 Monte Carlo steps (MCS) per site. Standard least squares fitting was then performed to fit to Eq. (9), with $x = 1$. In Fig. 1a we show typical results. The quality of the fits is excellent, and fitting to the $a(t)$ vs. t curve shown in the inset of Fig. 1a between $100 \leq t \leq 530$ MCS, we obtain

$$\beta(2) = 0.3324(7) \quad (12)$$

To compare with the standard method of obtaining β , we also calculated the surface width $w(t)$. The analysis of the data is conveniently aided by defining running exponents, which describe local variations in the data:

$$\beta_t \equiv \frac{\log[\bar{G}(t+n)] - \log[\bar{G}(t)]}{2 \log(n)} \quad (13)$$

where n can be a constant time step, or $n = t$.⁽⁹⁾ The definition for $w(t)$ is completely analogous. In Fig. 1b we show a comparison of the running exponents between the two quantities. As can clearly be seen, results from the width indeed tend to underestimate the true value of β , giving about 0.327(2). We also tested the effect of the convolution ansatz⁽⁹⁾ to both quantities, which improved results from the width to $\beta(2) = 0.330(3)$. Correspondingly, from $a(t)$ the result was $\beta(2) = 0.335(2)$.

To check further the quality of the fitting ansatz, we also monitored the behavior of the fitting parameter $\hat{\chi}_t$ as a function of time, as shown in Fig. 1c. After an initial transient there is a plateau which coincides with the best scaling regime for $a(t)$. Averaging over this region gives $\hat{\chi}(2) = 0.498(5)$, which is very close to the exact value of $1/2$. Similarly, estimating $\hat{z}(2)$ with $c = 0.9$ gives 1.503(2), and thus $\beta(2) \approx \hat{\chi}(2)/\hat{z}(2) \approx 0.331$. In addition, the Galilean invariance relation is satisfied very accurately, with

$$\hat{\chi}(2) + \hat{z}(2) \approx 2.001 \quad (14)$$

We note that using the standard methods of obtaining the scaling exponents, much larger system sizes are needed for comparable accuracy of the results.

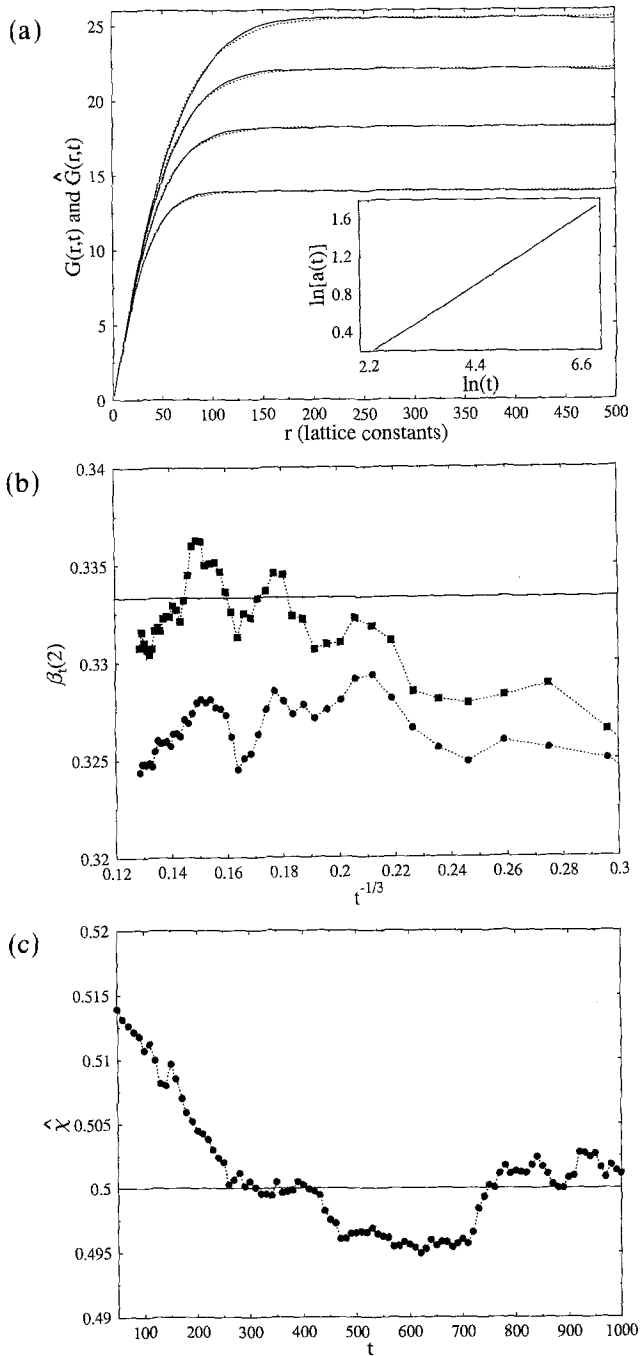


Fig. 1. (a) Correlation function $G(r, t)$ (solid lines) and fitted function $\hat{G}(r, t)$ for an $L = 3000$, two-dimensional GRSOS model at $t = 300, 500, 700,$ and 900 MCS/site. Only part of the functions is shown here. Inset shows $\ln[a(t)]$ vs. $\ln(t)$ as obtained from the fits. (b) Comparison between running exponents $\beta_i(2)$ of Eq. (13), as obtained from $w(t)$ (circles) and $a(t)$ (squares). Averaging over these gives $\beta(2) = 0.327(2)$ and $0.3337(7)$, respectively. Solid line denotes the exact value of $1/3$. (c) The running fitting parameter $\hat{\chi}(2)$ vs. time. Solid line denotes the exact value of $\chi(2) = 1/2$.

4.2. $d = 3 + 1$

In four dimensions, we performed extensive simulations of systems of several sizes: $L = 64, 100, 150, 190,$ and 250 . We shall first describe our most accurate results, which are for $L = 100$, where both the correlation function and the width were averaged over 2400 independent runs of up to 400 MCS/site, with additional test runs up to 800 MCS. In Fig. 2a we display results obtained from using the fitting ansatz with $x = 1/2$, while Fig. 2b shows running exponents [from Eq. (13)] from $a(t)$. The quality of the correlation function fits is again excellent, and a least squares fitting procedure between $120 \leq t \leq 350$ MCS gives

$$\beta(4) = 0.180(2) \quad (15)$$

which is also the result of averaging over the running exponents of Fig. 2b. Furthermore, the fitting parameter $\hat{\chi}_t$ in Fig. 2c again displays a plateau, from which $\hat{\chi}(4) = 0.294(3)$. To obtain $\hat{z}(4)$ as well, we carefully checked its dependence on c for $0.80 \leq c \leq 0.99$. An average over $c = 0.85, 0.9$ and 0.95 gives $1.709(9)$, from which the invariance relation becomes

$$\hat{\chi}(4) + \hat{z}(4) \approx 2.003 \quad (16)$$

This high-precision check on the Galilean invariance relation strongly supports the consistency of the fitting ansatz in these higher dimensions. However, estimating $\beta(4) \approx \hat{\chi}(4)/\hat{z}(4) \approx 0.172$ demonstrates that $\hat{\chi}(4)$ *underestimates* the real $\chi(4)$, as we shall directly show below.

For an independent consistency check of our results, we also determined $\chi(4)$ directly by calculating the width $w(L) \sim L^\chi$ in the *saturated regime* of $t \gg L^z$ for $L = 5, 10, 15, 20, 25, 30, 35, 40, 50,$ and 60 . Averaging over independent configurations was done until the results seemed to converge, with error bars estimated from variations between consecutive runs. Results for $w(L)$ are shown in Fig. 3. Despite considerable efforts, fluctuations in the data for the larger systems (especially for $L = 60$) remained, which demonstrates the difficulty of a precise determination of χ with this method. Nevertheless, for small system sizes we were able to assess the finite-size effect. Namely, $\chi(4)$ starts out near the value of 0.300 , increasing with system size up to about 0.31 . Our best estimate from a simple least squares fit for $20 \leq L \leq 50$ is

$$\chi(4) \approx 0.308(2) \quad (17)$$

which, however, does not include finite-size effects in a systematic manner. Thus, the error bar is perhaps not realistic. Nevertheless, this result is considerably larger than that obtained for $\hat{\chi}(4)$, demonstrating the approximate nature of the fitting ansatz (9). However, using the direct

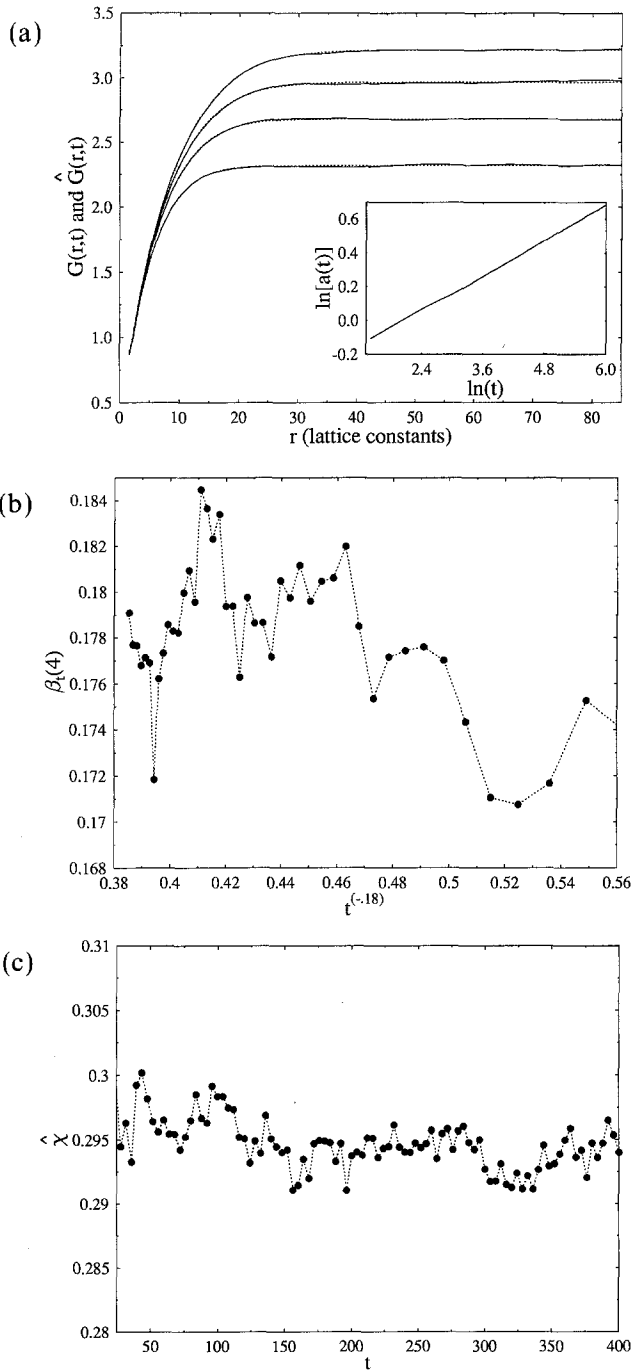


Fig. 2. $G(r, t)$ (solid lines) and $\hat{G}(r, t)$ at $t=90, 180, 270,$ and 360 MCS/site for the 100^3 GRSOS model. Inset shows $\ln[a(t)]$ vs. $\ln(t)$. Only part of the functions is shown. (b) Running exponents [Eq. (13)] from $a(t)$, averaging over which gives $\beta(4)=0.180(6)$. (c) The running parameter $\chi_1(4)$ vs. time.

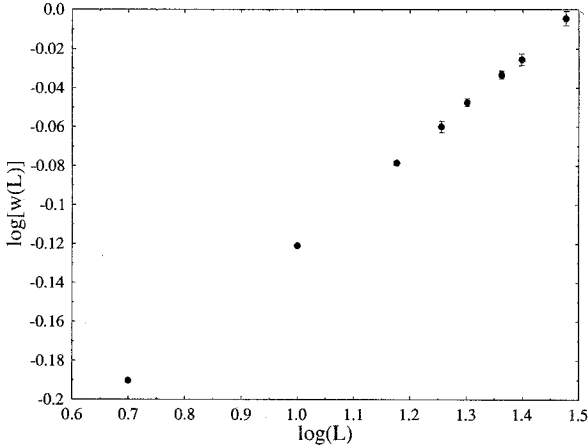


Fig. 3. Data for the saturated interface width $w(L)$ in four dimensions.

estimate of $\chi(4)$ above together with the Galilean invariance relation yields $\beta(4) \approx 0.182$, which is now fully consistent with the correlation function data of Eq. (15).

Despite the apparent consistency of the results presented above, we cannot completely rule out the influence of finite-size effects on the value of $\beta(4)$. To further study this question, we performed additional simulations for $L = 64$ (3000 runs), $L = 150$ (75 runs), $L = 190$ (100 runs), and $L = 250$ (two runs). The correlation function fits were performed for $L = 64, 150$, and 190 ; for $L = 250$ only the time-dependent width was computed. Except for the smallest system, for which the running exponents approach the value of 0.18 from below⁽¹⁵⁾ (see also Fig. 5), the data from $\bar{G}(t)$ were simply not good enough for an accurate determination of $\beta(4)$. Nevertheless, estimating $a(t)$ between $50 \leq t \leq 400$ MCS from a fit to the $L = 190$ data gives $\beta(4) = 0.185 \pm 0.013$, which is consistent with the above results. However, we can obtain a better grip on the systematic finite-size effects through the direct dependence of the slope of the width $w(t)$ on L . In Fig. 4a we show a comparison of $w(t)$ for all the system sizes studied here, with running exponents for $L = 100$ shown in Fig. 4b. Using least squares fits for each case, we can summarize the results as

$$\beta(4) = \begin{cases} 0.173(1) & \text{for } L = 64 \\ 0.1762(2) & \text{for } L = 100 \\ 0.179(5) & \text{for } L = 150 \\ 0.180(5) & \text{for } L = 190 \\ 0.182(1) & \text{for } L = 250 \end{cases} \quad (18)$$

The details of the fits are as follows: for $L=64$, $20 \leq t \leq 150$ MCS; for $L=100$, $50 \leq t \leq 350$ MCS; for $L=150$, the result is from two fits for $25 \leq t \leq 400$ MCS; for $L=190$, fitting for $50 \leq t \leq 400$ MCS and $400 \leq t \leq 840$ MCS yields $\beta(4) = 0.177(1)$ and $0.183(2)$, respectively, the average of which is 0.180 ; and finally for $L=250$, $30 \leq t \leq 320$ MCS, where the best scaling regime was found. Remarkably enough, at least for the range of system sizes studied here, the scaling exponent $\beta(4)$ seems to saturate around 0.18 , which is again fully consistent with $0.180(2)$ as obtained from the $L=100$ system using the fitting ansatz. We must note, however, that

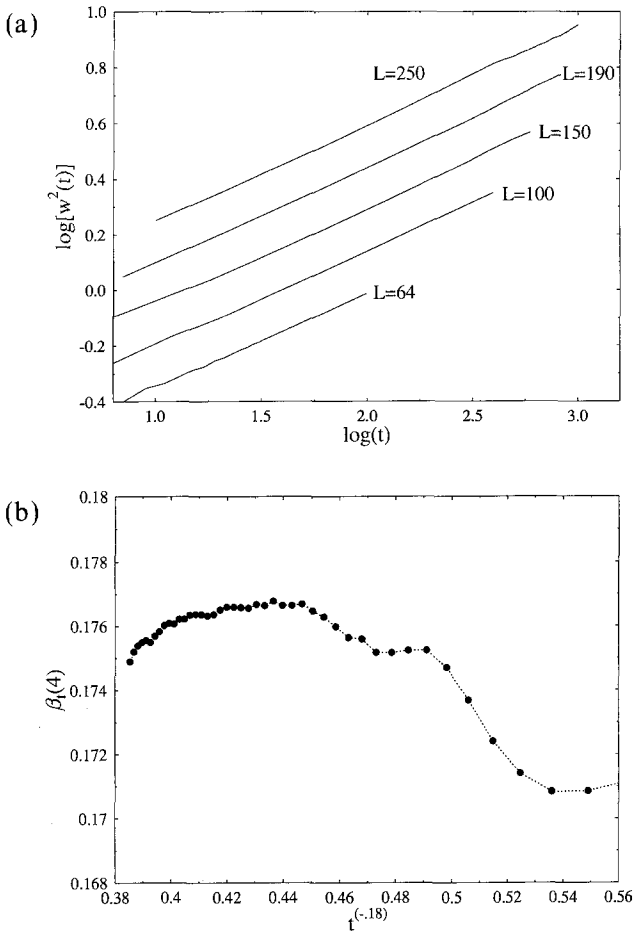


Fig. 4. The surface width for all four-dimensional systems studied here. The curves have been shifted for clarity. (b) Running exponents $\beta_t(4)$ from the width [Eq. (13)] for $L=100$. Averaging over these yields $\beta(4) = 0.1762(2)$.

there are significant fluctuations in the data of the largest systems for $w(L, t)$, which is reflected in the sensitivity of the value of $\beta(4)$ to the choice of the fitting regime. Thus, the numbers quoted in Eq. (18) beyond $L = 100$ should not be taken as reliable quantitative estimates, but rather as supporting our best estimate of $\beta(4) = 0.180(2)$. Nevertheless, if we use the data of Eq. (18) and assume that the usual $1/L$ finite-size scaling holds, an extrapolation to $1/L \rightarrow 0$ gives a slightly larger value of $\beta(4) \approx 0.184$.

In analogy to the two-dimensional case, we also attempted to use the convolution ansatz for improving results obtained from the widths. Rather surprisingly, however, this did not yield any improvements—e.g., for our best data for $L = 100$, $\beta(4) = 0.17(1)$ was obtained, with rather poor accuracy. Similarly, for $L = 190$, $\beta(4) \approx 0.185$. Our results clearly indicate that even up to $L = 190$, there are additional time-dependent terms present in $w^2(t)$ besides a constant “intrinsic width.”⁽⁹⁾

The remarkable feature of the result (15) is that it is considerably *smaller* than that obtained by Kim *et al.*,⁽¹⁴⁾ who estimated $\beta(4) = 0.20$ using a 64^3 system. The essential difference between their results and ours lies in the definition of time, for which they used the average height $\bar{h}(t)$ instead of the Monte Carlo time t . Namely, it has been shown that there is a finite-size correction to $\bar{h}(t)$, such that it only becomes directly proportional to time in the thermodynamic limit.⁽²⁸⁾ To compare our results directly with Kim *et al.*,⁽¹⁴⁾ we analyzed carefully our data for the 64^3 system.⁽¹⁵⁾ In Fig. 5 we summarize the results of this analysis, depicting

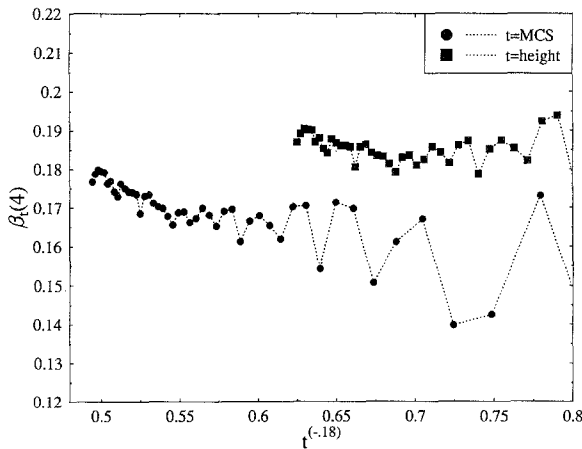


Fig. 5. Running exponents from $a(t)$ [Eq. (13)] for a 64^3 system using Monte Carlo time (circles) and the average height $\bar{h}(t)$ as definitions of time.

running exponents from $a(t)$ [again from Eq. (13)] using either MCS or $\bar{h}(t)$ as a definition of time. The latter clearly yields a larger value for $\beta(4) \approx 0.19$, which may explain the reason for the discrepancy.

4.3. $d = 4 + 1$

In five dimensions, our analysis parallels that of the four-dimensional case. However, only three system sizes were considered: $L = 20$ (3500 runs), $L = 50$ (100 runs), and $L = 70$ (three runs). The analysis of the correlation

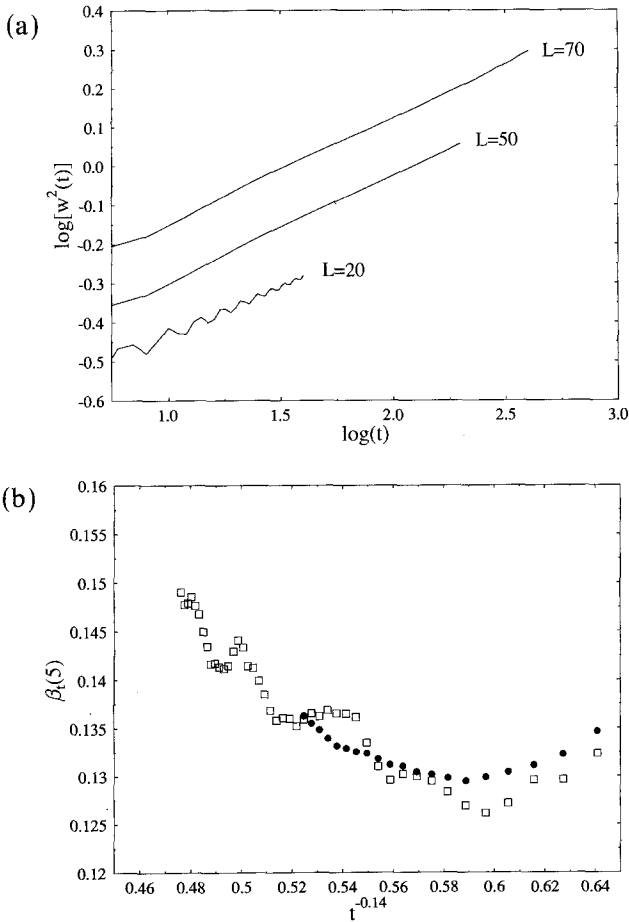


Fig. 6. (a) The surface width for all three five-dimensional systems. The curves have been shifted for clarity. (b) Running exponents [from Eq. (13)] for $L = 50$ (filled circles) and $L = 70$.

functions proved not to be feasible, due to large fluctuations in the data. Thus, we determined $\beta(5)$ simply by a least squares fit to $w(t)$, which is shown in Fig. 6a. The results are

$$\beta(5) = \begin{cases} 0.124(8) & \text{for } L = 20 \\ 0.135(2) & \text{for } L = 50 \\ 0.139(2) & \text{for } L = 70 \end{cases} \quad (19)$$

Since there is a clear dependence of $\beta(5)$ on system size, we also analyzed the behavior of the running exponents for $L = 50$ and 70 , which are shown in Fig. 6b. For $L = 70$, the result quoted above comes from $50 \leq t \leq 400$ MCS; however, due to a big fluctuation clearly visible in Fig. 6, somewhat larger values of the running exponents are obtained at the latest times, e.g., $\beta(5) = 0.150(1)$ for $200 \leq t \leq 400$ MCS. Thus we cannot pinpoint the value of $\beta(5)$ as accurately as for lower dimensions.

As a consistency check, we also calculated the size-dependent saturated width $w(L)$, which is shown in Fig. 7 for $L = 5, 10, 15, 18, 20, 23, 35,$ and 30 . The fluctuations in the data are clearly visible, but by simply fitting for the six largest systems we estimate

$$\chi(5) \approx 0.245(1) \quad (20)$$

Using Galilean invariance, this gives $\beta(4) \approx 0.140$, which agrees quite well with the estimate obtained from the width using the largest system size

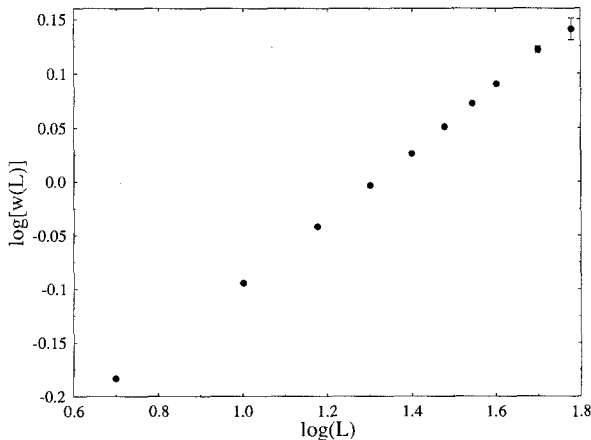


Fig. 7. Data for the saturated interface width $w(L)$ in five dimensions.

$L=70$. However, it is clear that our analysis here cannot exclude the influence of systematic finite-size effects, which may further increase the value of $\beta(5)$.

4.4. $d=5+1$, $6+1$, and $7+1$

It is already clear from the results of the previous section that increasing the substrate dimension further leads to severe finite-size effects. However, we have undertaken an effort to obtain at least *lower bound estimates* for the scaling exponent $\beta(d)$ above five dimensions. Besides being of fundamental interest in themselves, these estimates provide a check on the existence of a possible upper critical dimensionality d_u for the kinetic roughening problem. Namely, even for relatively small system sizes it should be possible to distinguish between a power law behavior of the width as opposed to logarithmic behavior expected above d_u .

The simulations for $d=5+1$, $6+1$, and $7+1$ were done for $L=30$, 17, and 11, respectively, with three independent runs for each case. In Fig. 8 we summarize the results for $w(t)$. Clear oscillations due to the layerwise growth are clearly visible for these relatively small systems. However, in each case a rather well-defined power law behavior can be seen, and using least squares fitting, we obtain the following estimates:

$$\beta(d) \gtrsim \begin{cases} 0.107(2) & \text{for } d=5+1 \\ 0.10(2) & \text{for } d=6+1 \\ 0.08(2) & \text{for } d=7+1 \end{cases} \quad (21)$$

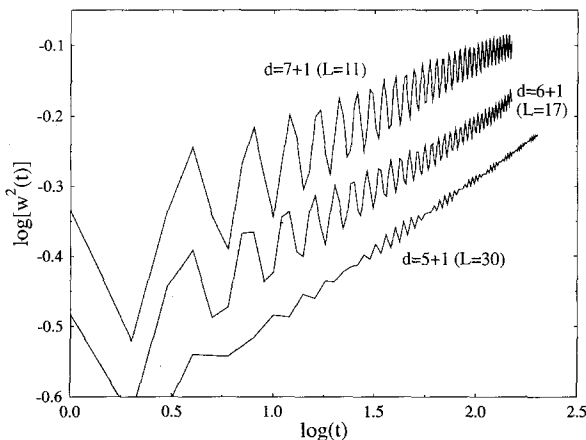


Fig. 8. The time dependence of the surface width for systems above five dimensions. Curves have been shifted for clarity.

Our results thus suggest that if a finite d_u exists, it has to be larger than eight.

5. SUMMARY AND DISCUSSION

To summarize, the purpose of this work has been to address three basic questions regarding the scaling exponents and universality within kinetic roughening, as described by the strong-coupling behavior of the KPZ equation. First, using a new fitting ansatz method, we have obtained quantitative estimates for the scaling exponents in four and five dimensions, which disagree with all existing theories and conjectures. Second, since our best estimate $\beta(4) = 0.180(2)$ is in excellent agreement with the results of Forrest *et al.*⁽⁴⁾ for a different growth model, these numbers themselves should be universal (see also Fig. 9). Why the recent directed polymer simulations of Kim *et al.*⁽²⁷⁾ differ from these new results is still an open question. Third, we have done additional simulations up to eight dimensions, finding no evidence for an upper critical dimension in the problem.

Finally, we would like to discuss briefly the application of the fitting ansatz (9) to the determination of universal amplitude ratios of the growth model. The amplitudes are usually defined through the steady-state correlation function⁽¹⁶⁾

$$C(\mathbf{r}, t) \equiv \lim_{t_0 \rightarrow \infty} \langle [\delta h(\mathbf{r} + \mathbf{r}_0, t + t_0) - \delta h(\mathbf{r}_0, t_0)]^2 \rangle \quad (22)$$

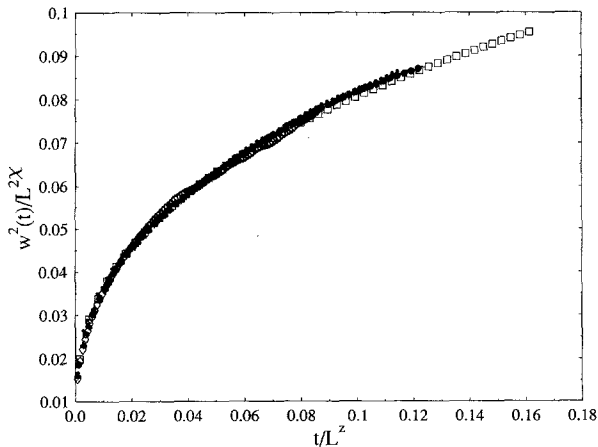


Fig. 9. Scaling of the interface width at $d = 3 + 1$, with $\beta(4) = 0.180$. System sizes are $L = 64, 100, 150, 190,$ and 250 . Scaling is much worse if $\beta(4) = 1/5$ from Eq. (5) is used.

where $\delta h \equiv h - \bar{h}$, and for which $C(r=0, t \rightarrow \infty) = Bt^{2B}$ and $C(r \rightarrow \infty, t=0) = Ar^{2x}$. The universal crossover scale is defined as the ratio⁽¹⁶⁾

$$g^* = \frac{\lambda}{2} \left(\frac{A}{B^{z/2}} \right)^{1/x} \quad (23)$$

The (nonuniversal) amplitudes A and B , along with λ , have recently been numerically (and analytically) determined for a variety of models in two dimensions, including the GRSOS model.⁽¹⁷⁻¹⁹⁾ Although Eq. (22) is *not* the correlation function we have calculated in this work, we can use the ansatz (9) to estimate A by choosing $1 \ll t_0 \ll L^z$ and approximating

$$G(r, t_0) \approx \hat{G}(r, t_0) \approx a(t_0) b(t_0) r^{2x} \quad (24)$$

for $r \ll t_0^{1/z}$. Thus, $A \approx a(t_0) b(t_0)$, assuming the product ab remains at least approximately constant within the scaling regime. In two dimensions, we have checked this to be true to a good approximation, and data for the $\hat{L} = 3000$ system give $A \approx 0.71_{-0.06}^{+0.04}$, which is in reasonable good agreement with the numerical result $A = 0.81$ of Krug *et al.*⁽¹⁷⁾ from much larger systems. We note that for our small system we expect the finite-size effect for the amplitude to be significant. For reference, we also determined $A(d)$ for $d = 3 + 1$ and $4 + 1$. In the former case, ab is roughly independent of system size beyond $L = 100$, which gives $A(4) \approx 0.19_{-0.04}^{+0.07}$, while the single $L = 50$ system gives $A(5) \approx 0.21_{-0.03}^{+0.04}$, the latter most likely an overestimate due to the finite-size effect. In both cases the product ab slightly decreases as a function of time. In these higher dimensions neither λ nor B has been determined. A more detailed discussion of these amplitudes is beyond the scope of this work, but would be desirable to probe further the issue of universality between different models.

ACKNOWLEDGMENTS

We acknowledge the Scientific Computer Centre of Finland for a generous allocation of computer time, and Asko Sainio for his technical help with the Cray XM-P. Prof. Martin Grant is also acknowledged for fruitful discussions. This research has been supported by the Academy of Finland. J.M.K. was supported in part by NSF grant DMR 89-18358 and wishes to acknowledge the Research Institute for Theoretical Physics at the University of Helsinki for its hospitality during his visit.

NOTE ADDED IN PROOF

For $L = 250$, we have improved our estimate for $\beta(4) = 0.178 \pm 0.01$ from the surface width. Using this new result an extrapolation to $1/L \rightarrow 0$ gives $\beta(4) \simeq 0.180$.

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